Reanalysis of the Froissart theorem

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”In Search for Fundamental Symmetries”
(Novozhilov-90) 2 – 4 December, 2014

The Froissart theorem (or Froissart bound) is known since 1961, after the paper

M.Froissart, Phys.Rev. **123** (1961) 1053.

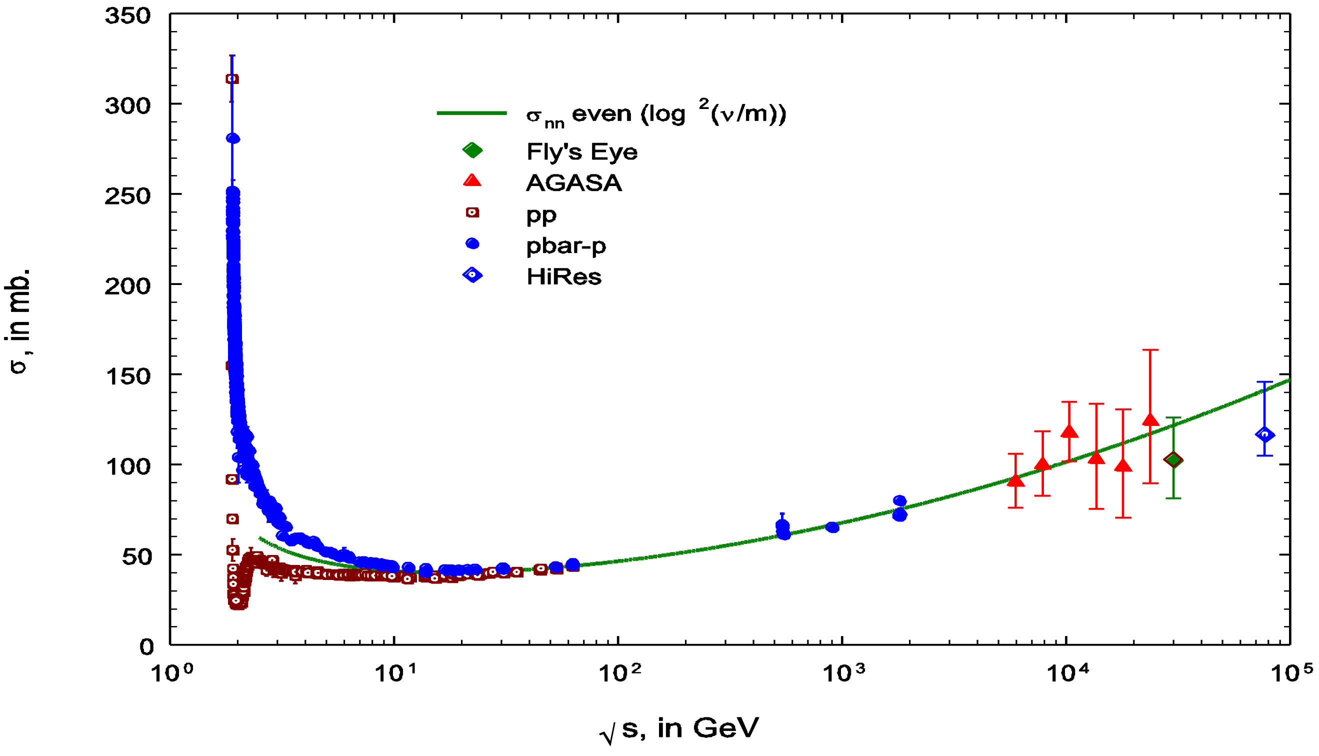
It seems to be most famous in High En. Phys., but its origin seems to be badly understood.

This theorem states that if the total cross section of strong interaction does increase with energy, then it cannot increase faster than $(\log E)^2$. (?)

The familiar opinion in the literature **:** violation of the Froissart bound would mean violation of unitarity. (?)

What did experimental data say before LHC ?

PreLHC measurements looked <u>consistent</u> with the expectation,
especially for nucleon-(anti)nucleon scattering,
where the highest energies could be reached, due to cosmic rays .

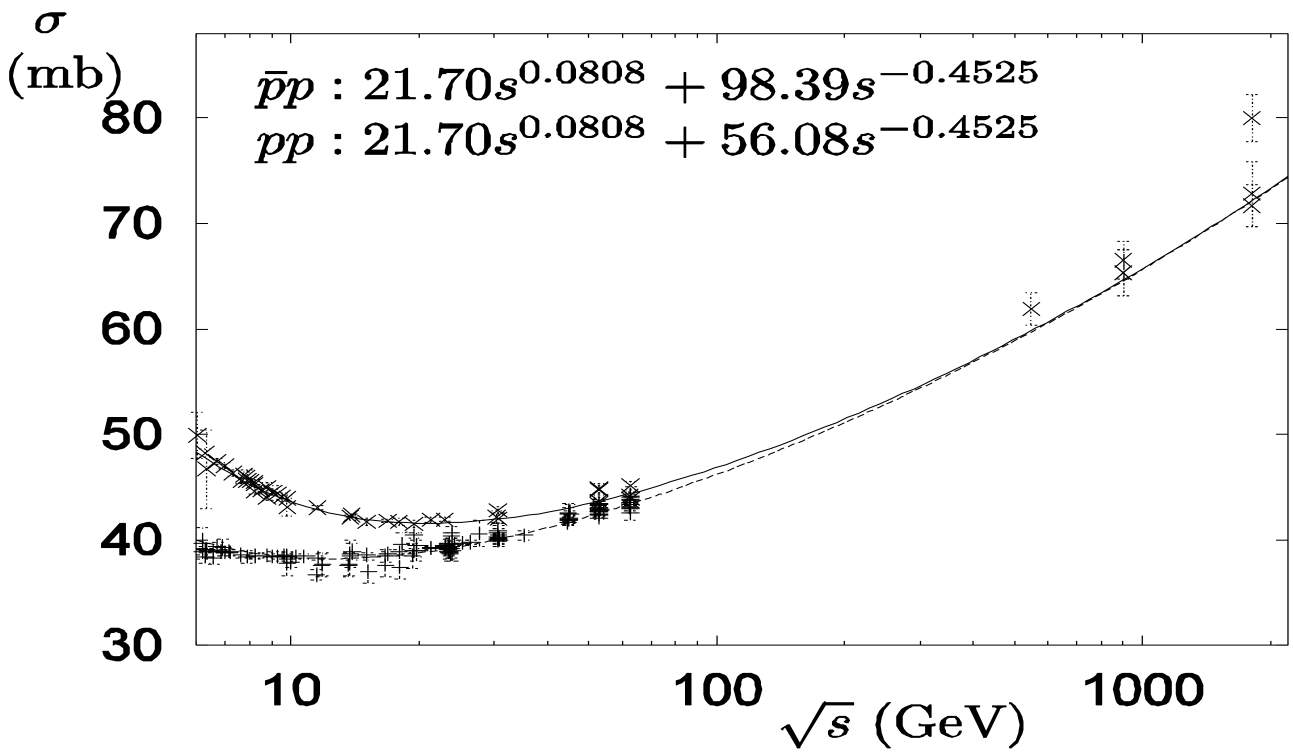


M.Block,
arXiv:1009.0313

Log^2 looks good

However,

large uncertainties of cosmic rays measurements enabled "heretic" descriptions of data, with ***faster*** (power) energy increase.



P.V.Landshoff, arXiv:0811.0260

It is reasonable, therefore,
to reconsider the theoretical background:

- Why, at all, quantum theory may provide <u>any</u> bounds (the Froissart bounds (F.b.) in particular) for energy increase of amplitudes and cross sections ?

- What asymptotics is preferred by the first principles ?
- How does the specific bound $\sim(\log E)^2$ arise ?
- What is the role of unitarity ?

Froissart *used* the assumptions:

a) unitarity condition;

b) strong interactions as an object to discuss;

c) elastic amplitudes satisfy

the double-spectral (Mandelstam) representation or,

at least,

the dispersion relation in the momentum transfer

(the both have never been proved for relativistic theories !).

Froissart *deduced* restrictions for

a) forward/backward amplitudes ($< s \ln^2\{s\}$)

and total cross sections ($< \ln^2\{s\}$);

b) fixed-angle amplitudes ($< s^{3/4} \ln^{3/2}\{s\} / \sin^{1/2}\theta$),

this slower energy growth necessitates the forward peak.

Main original steps to the Froissart theorem

- $A(s,t) = \sum_\ell (2\ell+1)\, a_\ell(s)\, P_\ell(z)$, $t=2k^2(-1+z)$, $z=\cos\theta$;
- $|a_\ell| \le 1$ at any ℓ - unitarity;
- $|a_\ell|$ at large ℓ decreases: $< B(s) \exp(-\ell\,\alpha_0)$ -

  it is a consequence of the Gribov-Froissart representation (integrals along cuts of dispersion relations in momentum transfer),

  $t_0=2k^2(-1+\mathrm{ch}\,\alpha_0)$ – the nearest t-channel singularity (at high energy $\alpha_0^2 \sim s^{-1}$ for fixed t_0);
- the two boundaries intersect at $\ell=L$;
- important at high s is only the sum over $\ell \le L$;

  thus, the high-energy bound for $A(s,t)$

  is determined by behavior of L ;

Main original steps to the Froissart theorem

- for the forward amplitude:

  $|A(s,0)| < \sum_{\ell} (2\ell +1) \approx L^{2}$;

- for the fixed-angle amplitude with $z = \cos\theta$:

  $|A(s,z)| < \sum_{\ell} (2\ell +1)/(\ell \sin\theta)^{1/2} \approx C\, L^{3/2} /(\sin\theta)^{1/2}$;

  (note essential difference in asymptotics for the forward and nonforward amplitudes, though $\theta=0$ is <u>not</u> a real singularity).

- What is the value of L ?

  $1 = B(s) \exp(-L\, \alpha_{0}),\ \ \alpha_0 \sim s^{-1/2},\ \ L = \alpha_0^{-1} \ln B(s).$

- If $B(s) \sim (s/s_0)^N$ with fixed s_0 , then $L \sim s^{1/2} \ln\{s/s_0\}$

  (N is related to the number of subtractions in disp. relations);

- for the forward amplitude:

  $|A(s,0)| < C\, s \ln^2(s/s_0)$, **!** $\sigma_{tot} < C \ln^2(s/s_0)$ **!** .

  (C and s_0 are not fixed theoretically !)

Modified approach to the Froissart bounds

(YaA, Phys.Rev. D**84** (2011) 056012; arXiv:1104.5314)

assumes:

a) **unitarity** condition (as originally);

b) **no** explicit assumptions on **kind of interactions**;

c) instead, only **absence of massless particles**

(this marks the strong interactions !);

d) **no** assumptions on **dispersion relations**;

deduces limitations for

a) forward/backward amplitudes ($< s \ln^2\{B_0(s)\}$)

and total cross sections ($< \ln^2\{B_0(s)\}$);

b) fixed-angle amplitudes ($< s^{3/4} \ln^{3/2}\{B_0(s)\}$),

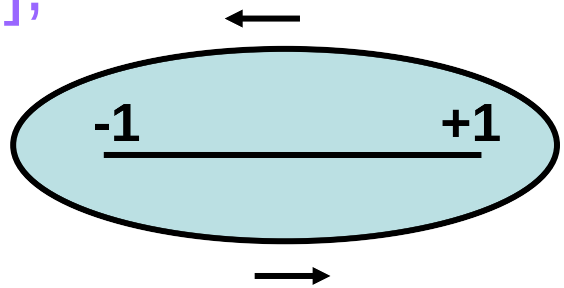
$B_0(s)$ is determined by the amplitude $A(s,z)$ at nonphysical z.

Modified derivation of F.b.

$$a_\ell = \tfrac{1}{2} \int_{-1}^{+1} dx' \, P_\ell(x') \, A(s,x') \; ;$$

$$\tfrac{1}{2} \pi \, P_\ell(x') = \tfrac{1}{2} \, i \, [Q_\ell(x'+i\varepsilon) - Q_\ell(x'-i\varepsilon)];$$

$$a_\ell = - \, i \, /(2\pi) \oint dz' \, Q_\ell(z') \, A(s,z') \; ;$$



It could be the way to the Gribov-Froissart representation.
But let us stop here.

We can choose the contour to be an ellipse
with $z' = \mathrm{ch}\,(\alpha + i\,\varphi')$, $\alpha = \mathrm{const} > 0$, $-i\,dz' = \mathrm{sh}(\alpha + i\varphi')\,d\varphi'$;
it may be blown up, until it touches a singularity,
i.e. $\alpha = \alpha_0$ (>0; in the physical region $\alpha = 0$).

Modified derivation of F.b.

On the contour, at large ℓ (the same inequality as used by Froissart)

$$|\mathrm{sh}(\alpha+i\varphi')\, Q_\ell(z')| < \exp[-\alpha(\ell+½)]\, (\pi\, \mathrm{ch}\, \alpha)^{½},$$

independent of φ' (may be pulled out from the integral).

Therefore, at high energies and large ℓ

$$|a_\ell(s)| < B_0(s)\exp(-\ell\,\alpha_0) ;$$

$B_0(s)$ is determined by the nonphysical amplitude $A(s,z')$, with z' on the integration contour.

Again, we have the "critical" value L and the whole set of F.b. (in terms of L):

$$\exp(L\,\alpha_0) \approx B_0(s), \quad L \sim s^{½} \ln\{B_0(s)\} .$$

The "canonical" bounds appear only under an **additional hypothesis** (independent of unitarity !) : $B_0(s) \sim s^N$.

It is usually "hidden" in the dispersion relations with subtractions.

Some lessons from the past

- ***Power growth*** of an amplitude in the energy is
  a necessary input to write
  the **dispersion relation** **in energy,**
  together with ***singularity positions*****.**

- Those two inputs have different status **:**
  singular point positions are determined
  by unitarity,
  while the **power growth** has **never received**
  either physical or mathematical **justification**
  in relativistic theories.

Some lessons from the past

- The limited power growth
  (and Mandelstam representation)
  was prooved for the Shroedinger eqation
  with Yukawa-like potential
  (T.Regge, Nuovo Cim.**14** (1959) 951;
  18 (1960) 947).

  The proof is closely related with nonlinearity
  of Regge trajectories (having limited real parts).
- Empirical linearity of hadron Regge trajectories,
  with unlimited real parts, may give evidence
  for a faster (than $\sim \boldsymbol{s}^N$) growth of amplitudes
  (at least, in some nonphysical configurations)
  and, therefore, a faster (than $\sim \ln^2 \boldsymbol{s}$) growth
  of cross sections.

Enhancing the Froissart bounds

(YaA, Phys.Rev. D**84** (2011) 056012)

Froissart, for simplicity, used the inequality

$$|\mathrm{sh}(\alpha+i\varphi')\, Q_\ell(z')| < \exp[-\alpha(\ell+\tfrac{1}{2})]\,(\pi\,\mathrm{ch}\,\alpha\,)^{1/2}\,,$$

while the more exact one looks as

$$|\mathrm{sh}(\alpha+i\varphi')\, Q_\ell(z')| < \exp[-\alpha(\ell+\tfrac{1}{2})]\,(\pi\,\mathrm{ch}\,\alpha\,/\ell)^{1/2}\,.$$

If we apply it at high ℓ, then

$$|a_\ell(s)| < B_0(s)\exp(-\ell\,\alpha_0)/\,\ell^{1/2}\,;$$

Again, we have the "critical" value L :

$$\exp(L\,\alpha_0)\,L^{1/2} \approx B_0(s)\,,$$

but now it is smaller and provides a stronger restrictions.

May it be further restricted?

New Froissart-like bounds

(YaA, Phys.Rev. D **84** (2011) 056012)

Froissart derived bounds for amplitudes of fixed-angle scattering

$A(s,\cos\theta)$ at $\theta = 0$, or $\theta = \text{const} \neq 0$.

Similar bounds may be obtained for fixed-***t*** scattering

(recall that $\boldsymbol{\theta} \to \mathbf{0}$ at high energy and fixed ***t***,

so the ***t***-bounds cannot be derived directly from $\boldsymbol{\theta}$-bounds):

$$|A(s,t)| < s \ln^{3/2}\{B_0(s)\} / |t|^{1/4} \quad \text{for} \quad t < 0;$$

while, as before, $|A(s,0)| < s \ln^{2}\{B_0(s)\}$ for $t = 0$

(recall that ***t***=0 is not a singularity of the amplitude;
nevertheless, asymptotics breaks in this point).

Thus, the forward amplitude may grow with energy faster,
than nonforward one.

If it is so indeed, the small-***t*** peak should necessary evolve,
its slope ***b*** should grow with energy ..

New Froissart-like bounds

If, in addition, the forward amplitude and σ_{tot} are ***saturated***
(*i.e.,* grow maximally fast),
then the slope ***b*** should grow with energy **just as** σ_{tot} or **faster :**

the ratio $(\sigma_{tot} / b) \sim (\sigma_{el} / \sigma_{tot})$ should not increase !

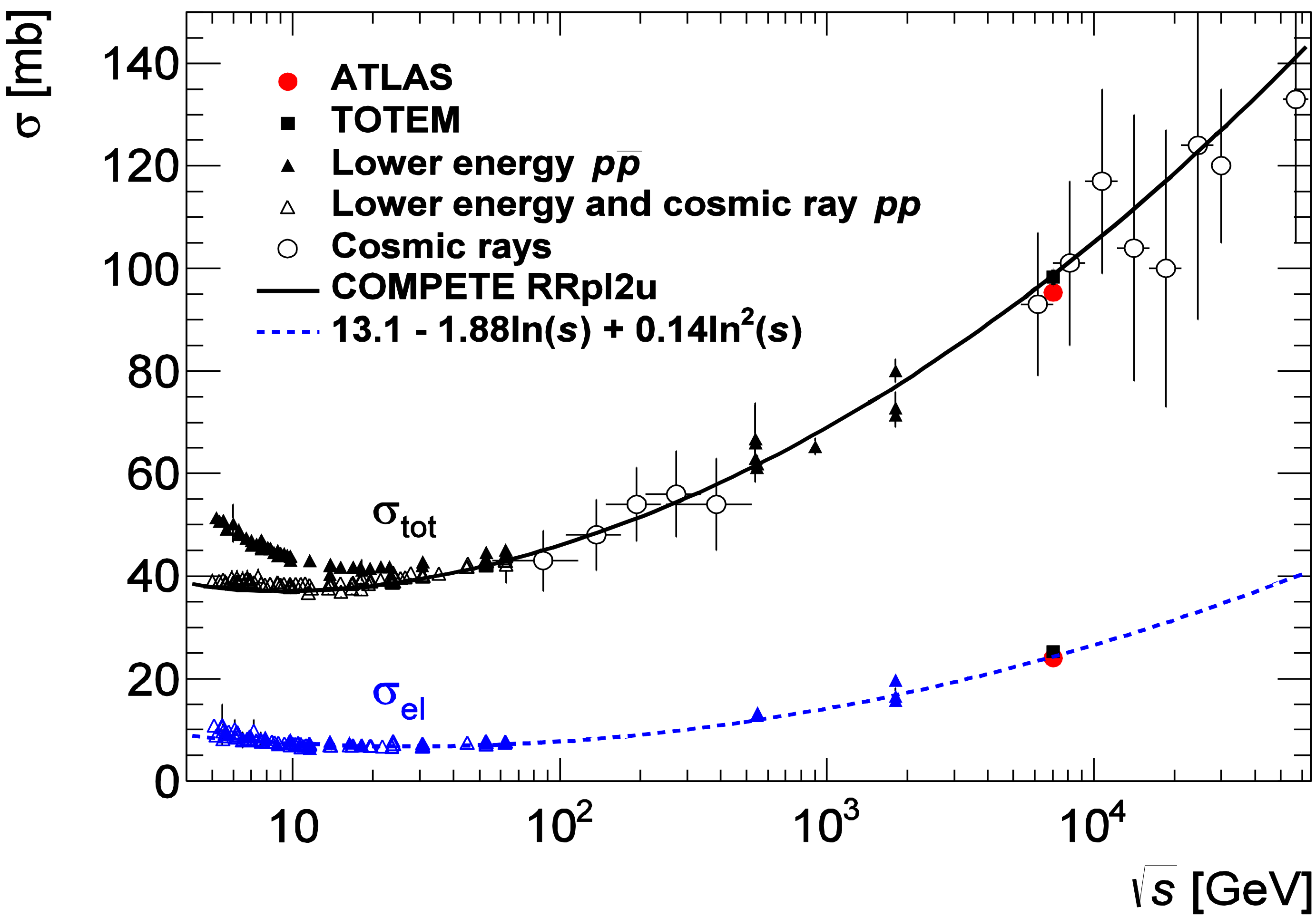
Violation of this property would mean violation
either of **unitarity,** or of **saturation** !

If the nonforward amplitude is also saturated, the ratio has to be constant.

The physical meaning is clear enough**:**
For an absolutely black disc, σ_{tot} can grow only as the geometrical area,
i.e., as $R^2 \sim \boldsymbol{b}$ **.**
The area may grow faster than σ_{tot} **,** if transparency increases.
Then the ratio (σ_{tot} / b) decreases and may tend to zero (or to a constant).
Increase of the ratio would mean the increasing blackness; it must finish somehow,
say, after reaching the absolute blackness (or earlier)**.**

What do current experimental data say?

Present cross section measurements,
including TOTEM and ATLAS @ LHC,
satisfy the fit having the log-squared asymptotics. .

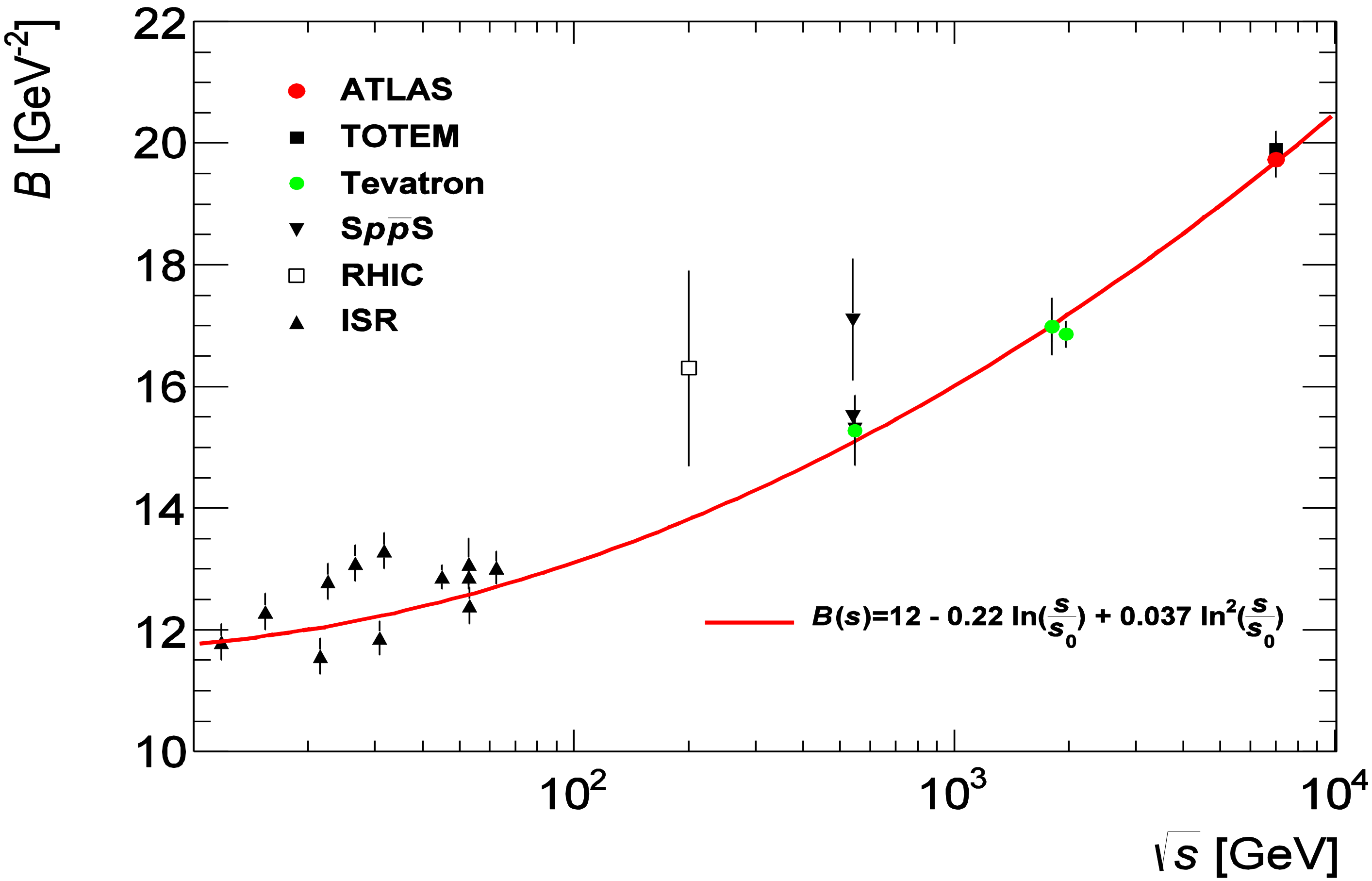


G.Antchev et al.
(TOTEM Col.),
Eur.Phys.Lett.
96 (2011) 21002;
arXiv:1110.1395

G.Aad et al;
(ATLAS Coll.)
Nucl.Phys. B
889 (2014) 486
arXiv:1408.5778

What do current experimental data say?

Present slope measurements,
including TOTEM and ATLAS @ LHC data ,
Also satisfy the fit having the log-squared asymptotics .



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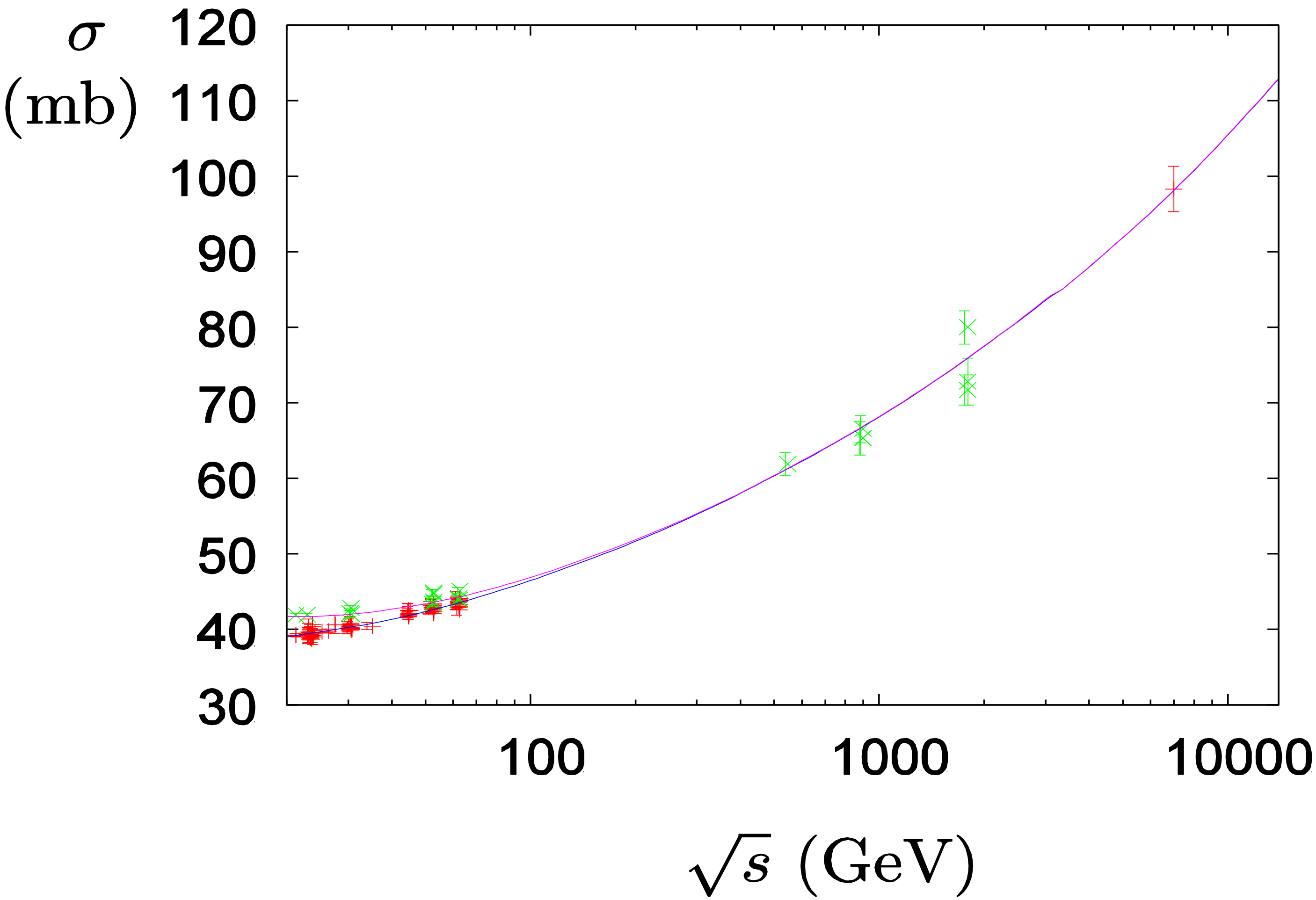
However,

"heretic" descriptions of data, e.g., with power asymptotics, are also possible, even after the first LHC results.

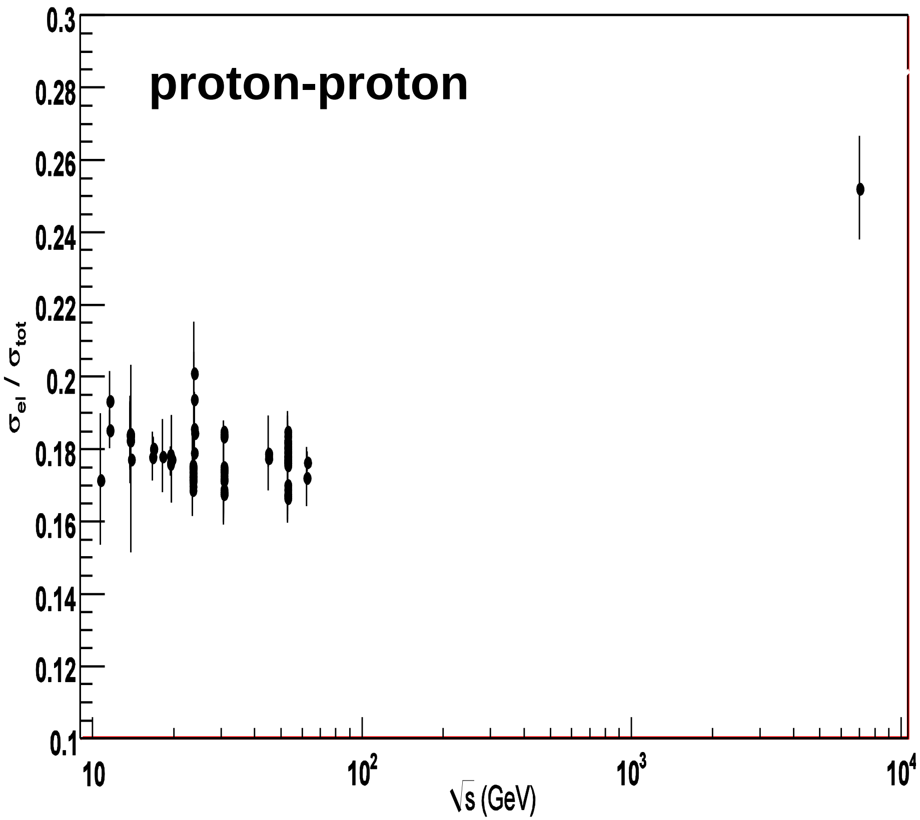
Will this be possible as well, when accelerator data expand to even higher energies ?



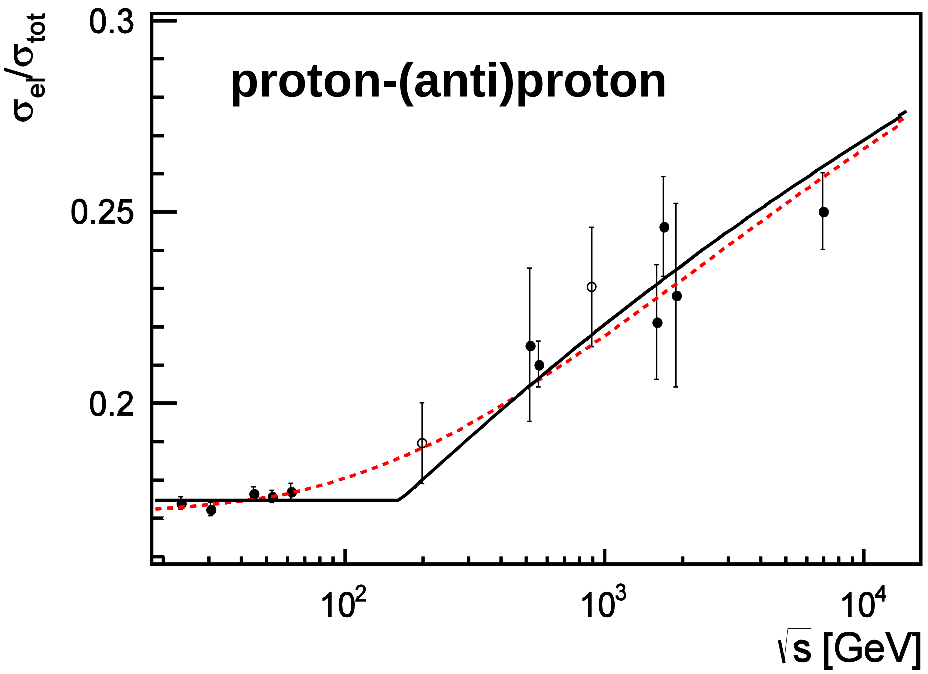
A.Donachi, P.V.Landshoff, arXiv:1112.2485

The ratio (σ_{el} / σ_{tot}) is nearly constant between 10 and 100 GeV,
but increases above 100 GeV,

either for proton-proton, or for proton-antiproton interactions
(including SPS, Fermilab, and TOTEM @ LHC data).



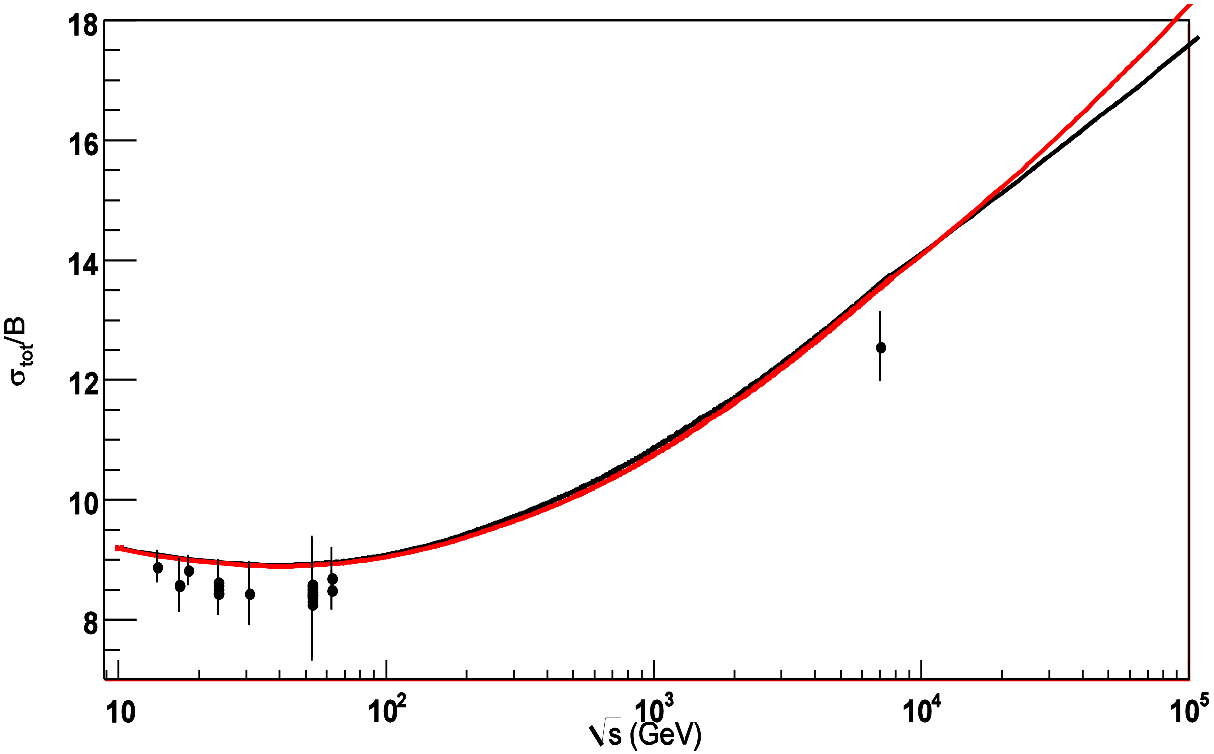
D.A.Fagundes, M.J.Menon,
NP A**880** (2012) 1; arXiv:1112.5115



R.Conceicao et al.,
arXiv:1107.0912

The same is true for the ratio (σ_{tot} / ***b***) .

This increase should finish after all,
if σ_{tot} is to be **saturated !**



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Conclusions

- The very general result, which is the <u>genuine meaning</u> of Froissart's theorem, is the much softer energy growth for physical amplitudes *vs*. nonphysical ones. (mainly, due to mathematical properties of the Legendre functions).

- Particular form of high-energy asymptotics of σ_{tot} is, theoretically, an open question.
  Commonly believed log-squared one is related to an additional suggestion **(<u>never justified !</u>)** of power growth for amplitude(s) in nonphysical configurations.

- Violation of log-squared behavior does not violate unitarity. It contradicts only to the additional asymptotic assumption.

- The familiar Froissart bounds may be enhanced.

Conclusions

- There are indirect arguments
  for cross sections growing with energy
  faster than the “canonical” log-squared one.

- The observed relation between σ_{tot}
  and the diffraction slope ***b***
  gives evidence that the observed energy growth
  is **not saturated.** The present behavior should
  somehow change !

- Further studies, both experimental and theoretical,
  are necessary
  (influence of spin, possible enhancements,
  combination of unitarity in cross-channels,…)